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Coherent States for Generalized oscillator with Finite-Dimensional Hilbert Space [1]

Abstract

The construction of oscillator-like systems connected with the given set of orthogonal polynomials and coherent states for such systems developed by authors is extended to the case of the systems with finite-dimensional state space. As example we consider the generalized oscillator connected with Krawtchouk polynomials.

1 Introduction

In the last years the interest to studying of the oscillator-like systems (so-called "generalized oscillators") and to using such systems in various areas of quantum mechanics ([1]-[3]) has significantly increased. In works [4]-[12] authors suggested a new approach to definition of generalized oscillators, connected with the given system of orthogonal polynomials and to construction of coherent states for these oscillators. Within the framework of this approach we investigated generalized oscillators, connected with classical orthogonal polynomials of a continuous argument (such as Laguerre, Legendre, Chebyshev, and Gegenbauer polynomials), orthogonal polynomials of a discrete argument (such as Meixner and Charlier polynomials) as well as systems connected with q-analogues of Hermite polynomials. All these oscillator-like systems defined in an infinite-dimensional state space. However many interesting applications in nonlinear quantum optics are connected with finite-dimensional state spaces [13]. In these considerations the Krawtchouk polynomials, which form the important class of classical orthogonal polynomials of discrete argument are frequently used. Convenience of these polynomials for applications is connected with the fact that these polynomials can be considered as a finite-dimensional approximation of the Hermite and Charlier polynomials [14]. The generalized oscillator connected with Krawtchouk polynomials [15]-[17] is a typical example of generalized oscillator in finite-dimensional Hilbert space.

Within the framework of our approach, we discuss here the construction of generalized oscillator in finite-dimensional Fock space and definition of its coherent states in a such way that in the limiting case when dimension of space goes to infinity these coherent states become ones of

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appropriate generalized oscillator with infinite state space[2]. In the finite-dimensional state space the annihilation operator has only one eigenvector with zero eigenvalue (the vacuum state), so the standard definition of Barut-Girardello coherent states as eigenstates of the annihilation operator is inapplicable. There are several variants of definition of coherent states for finite-dimensional analogue of the usual boson oscillator (the spin coherent states, phase coherent states etc.[13]). In present work we consider a generalization of the definition given in [18],[19] for the standard (boson like) FD-oscillator. This generalization is a modification of the Glauber definition [20] to the case of "truncated" exponential operators. The coherent states obtained by such construction can be considered as coherent states of Klauder - Gazeau type [21].

1. We recall, following [4], the basic steps of our construction of oscillator-like systems. For brevity we discuss here only a symmetric case, when the Jacobi matrix defined by recurrent relations for considered orthogonal polynomials has a zero diagonal (in this case an orthogonality measure is symmetric under reflection with respect to the origin of coordinates).

Let us denote by $\mathcal{H}_\mu$ the Hilbert space $L^2(\mathbb{R}; \mu)$, where μ is a positive symmetric probability Borel measure on the real axis. We suppose that all power moments of this measure are finite and

$$\mu_0 = \int_{-\infty}^{\infty} \mu(\mathrm{dx}) = 1, \qquad \mu_{2k+1} = \int_{-\infty}^{\infty} \mathrm{x}^{2k+1} \mu(\mathrm{dx}) = 0. \quad \mathrm{k} = 0, 1, \ldots. \tag{1}$$

We define a positive numerical sequence

$$\{b_n\}_{n=0}^{\infty}, \quad b_n > 0, \quad b_{-1} = 0$$

as the solution of the algebraic system

$$\sum_{m=0}^{[\![\frac{n}{2}]\!]} \sum_{s=0}^{[\![\frac{n}{2}]\!]} \frac{(-1)^{n+s}}{(b_{n-1}^2)!} \alpha_{2m-1,n-1} \alpha_{2s-1,n-1} \mu_{2(n-m-s+1)} = b_{n-1}^2 + b_n^2, \quad n \geq 0, \tag{2}$$

where

$$\mu_{2k} = \int_{-\infty}^{\infty} x^{2k} \mu(\mathrm{dx}), \quad \mathrm{k} = 0, 1, \ldots, \tag{3}$$

$$\alpha_{-1,-1} = 0, \qquad \alpha_{2p-1,n-1} = \sum_{k_1=2p-1}^{n-1} b_{k_1}^2 \sum_{k_2=2p-3}^{k_1-2} b_{k_2}^2 \cdot \ldots \cdot \sum_{k_p=1}^{k_{p-1}-2} b_{k_p}^2. \tag{4}$$

Here the symbol $[\![x]\!]$ - denotes the whole part of x, and "a factorial on an index" $(b_{n-1}^2)!$ is defined by relation $(b_{n-1}^2)! = b_0^2 b_1^2 \cdot \ldots \cdot b_{n-1}^2$.

It is easy to check, that the relations

$$b_0^2 = \mu_2, \quad b_1^2 = \frac{\mu_4}{\mu_2} - \mu_2, \ldots, \tag{5}$$

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[2]In the literature the oscillator with finite-dimensional state space is usually named - finite oscillator and we frequently called them FD-oscillators for brevity.

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give the unique solution of the system (2).

The system of polynomials $\{\psi_n(x)\}_{n=0}^{\infty}$ is called a canonical system associated with the measure μ, if the following recurrent relations are fulfilled

$$x\psi_n(x) = b_n\psi_{n+1}(x) + b_{n-1}\psi_{n-1}(x), \quad n \geq 0, \quad b_{-1} = 0, \tag{6}$$

$$\psi_0(x) = 1, \tag{7}$$

where the positive coefficients $\{b_n\}_{n=0}^{\infty}$ are solutions of the system (2).

heorem 1 (see [4]). *Let the system of polynomials $\{\psi_n(x)\}_{n=0}^{\infty}$ satisfies to the relations (6) and (7) with coefficients $\{b_n\}_{n=0}^{\infty}$, forming a positive sequence and let μ be some symmetric probability measure on $\mathbb{R}$. This system of polynomials is orthogonal with respect to the measure μ if and only if the polynomial system $\{\psi_n(x)\}_{n=0}^{\infty}$ is the canonical polynomial system associated with the measure μ (i.e. the coefficients $\{b_n\}_{n=0}^{\infty}$, are solutions of the system (2)).*

We define the selfadjointed operators: "coordinate" X_μ, "momentum" P_μ and quadratic Hamiltonian $H_\mu = X_\mu^2 + P_\mu^2$ by their action on elements of the basis $\{\psi_n(x)\}_{n=0}^{\infty}$ in the Hilbert space $\mathcal{H}_\mu$, according to formulas

$$X_\mu\psi_0(x) = b_0\psi_1(x), \quad X_\mu\psi_n(x) = b_n\psi_{n+1}(x) + b_{n-1}\psi_{n-1}(x), \quad n \geq 1, \tag{8}$$

$$P_\mu\psi_0(x) = -ib_0\psi_1(x), \quad P_\mu\psi_n(x) = i\left(b_{n-1}\psi_{n-1}(x) - b_n\psi_{n+1}(x)\right), \ n \geq 1, \tag{9}$$

$$H_\mu\psi_n(x) = \lambda_n\psi_n(x), \qquad n \geq 0, \tag{10}$$

where

$$\lambda_0 = 2b_0^2, \qquad \lambda_n = 2(b_{n-1}^2 + b_n^2), \qquad n \geq 1. \tag{11}$$

We define further the creation and annihilation operators by the standard relations

$$a_\mu^+ := \frac{1}{\sqrt{2}}\left(X_\mu + iP_\mu\right), \qquad a_\mu^- := \frac{1}{\sqrt{2}}\left(X_\mu - iP_\mu\right), \tag{12}$$

These operators act on the elements of the basis in Hilbert space $\mathcal{H}$, according to the relations

$$a_\mu^+\psi_n(x) = \sqrt{2}b_n\psi_{n+1}(x), \qquad a_\mu^-\psi_n(x) = \sqrt{2}b_{n-1}\psi_{n-1}(x), \quad n \geq 0, \quad b_{-1} = 0. \tag{13}$$

We shall consider Hilbert space $\mathcal{H}_\mu$ as a functional realization of the Fock space $\mathcal{F}$ with basis $\{|n\rangle\}_{n=0}^{\infty}$, so that $\{\psi_n(x)\}_{n=0}^{\infty}$ (where $\psi_n(x) = \langle x|n\rangle$) gives realization of this basis in "coordinate" representation. We define also the self-ajointed operator $\mathcal{N}$, "numbering" basic elements,

$$\mathcal{N}|n\rangle = n|n\rangle, \quad n \geq 0, \tag{14}$$

and the operator $B(\mathcal{N})$, acting on basic vectors, according to formulas

$$B(\mathcal{N})|n\rangle = b_{n-1}^2|n\rangle, \quad n \geq 0, \tag{15}$$

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so that

$$B(\mathcal{N} + I)|n\rangle = b_n^2|n\rangle, \quad n \geq 0, \tag{16}$$

Let us note that it is not supposed, that $\mathcal{N} = a_\mu^+ a_\mu^-$.

It is simple to check the validity of commutation relations

$$\left[a_\mu^-, a_\mu^+\right] = 2\left(B(\mathcal{N} + I) - B(\mathcal{N})\right), \qquad \left[\mathcal{N}, a_\mu^\pm\right] = \pm a_\mu^\pm. \tag{17}$$

We shall call an algebra $\mathcal{A}_\mu$, generated by operators $a_\mu^\pm$, $\mathcal{N}$ with commutation relations (17), the generalized oscillator connected with given system of polynomials $\{\psi_n(x)\}_{n=0}^\infty$ orthonormalized with respect to the symmetric probability measure μ. Below we shall use the symbol $\mathcal{A}_\mu$ to denote the generalized oscillator as well.

2. In a general case when the measure μ is not symmetric, it is possible to find from the given sequence of its moments $\{\psi_n(x)\}_{n=0}^\infty$, $(\mu_0 = 1)$ two uniquely defined sequences of real numbers $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ and define a canonical system of polynomials $\{\psi_n(x)\}_{n=0}^\infty$, which are orthogonal with respect to a measure μ and satisfy recurrent relations

$$x\psi_n(x) = b_n\psi_{n+1}(x) + a_n\psi_n(x) + b_{n-1}\psi_{n-1}(x), \qquad n \geq 0, \quad b_{-1} = 0, \tag{18}$$

with the initial condition

$$\psi_0(x) = 1. \tag{19}$$

Together with this system $\{\psi_n(x)\}_{n=0}^\infty$ of polynomials, orthogonal with respect to a measure μ, we shall consider one more system of polynomials $\left\{\psi_n^{(0)}(x)\right\}_{n=0}^\infty$, which are orthogonal with respect to another symmetric measure μ^0 and satisfy recurrent relations

$$x\psi_n^{(0)}(x) = b_n\psi_{n+1}^{(0)}(x) + b_{n-1}\psi_{n-1}^{(0)}(x), \qquad n \geq 0, \quad b_{-1} = 0, \tag{20}$$

with the initial condition

$$\psi_0(x) = 1. \tag{21}$$

Let us denote by $\mathcal{A}_{\mu^0}$ the generalized oscillator, constructed by the method described above in the section **1.** and defined in the space $\mathcal{F}_{\mu^0}\left(= \mathcal{H}_{\mu^0}\right)$.

To construct an oscillator $\mathcal{A}_\mu$, connected with the initial measure μ, it is necessary to introduce, following [4], the generalized coordinate operator $\widetilde{X}_\mu$, the generalized momentum operator $\widetilde{P}_\mu$ and Hamiltonian $\widetilde{H}_\mu$ as well as ladder creation and annihilation operators $\widetilde{a}_\mu^\pm$, according to formulas

$$\widetilde{X}_\mu = \mathrm{Re}\left(\mathrm{X}_\mu - \mathrm{P}_\mu\right), \quad \widetilde{P}_\mu = -\mathrm{iIm}\left(\mathrm{X}_\mu - \mathrm{P}_\mu\right), \quad \widetilde{H}_\mu = \widetilde{X}_\mu^2 + \widetilde{P}_\mu^2, \tag{22}$$

$$\widetilde{a}_\mu^\pm = \frac{1}{\sqrt{2}}\left(\widetilde{X}_\mu \pm \widetilde{P}_\mu\right), \tag{23}$$

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and to define the operator $\widetilde{\mathcal{N}}_\mu$. Operators X_μ and P_μ are defined by relations (8) and (9), but using the commutation relations (18) instead of (6) (see [4]).

It is follows from the results of work [4] that an action of operators $\widetilde{a}_\mu^\pm, \widetilde{\mathcal{N}}_\mu$ on the basic elements of Fock space $\mathcal{F}_\mu$ can be defined by the same relations as an action of operators $\widetilde{a}_{\mu^0}^\pm, \widetilde{\mathcal{N}}_{\mu^0}$ in the space $\mathcal{F}_{\mu^0}$, so operators $\widetilde{a}_\mu^\pm, \widetilde{\mathcal{N}}_\mu$ satisfy in space $\mathcal{F}_\mu$ the same commutation relations (17) as operators $\widetilde{a}_{\mu^0}^\pm, \widetilde{\mathcal{N}}_{\mu^0}$ in $\mathcal{F}_{\mu^0}$. Hence, $\mathcal{A}_\mu$ and $\mathcal{A}_{\mu^0}$ define unitary equivalent representations of the same generalized oscillator algebra.

3. In works [13], [18], [19] were considered oscillators, with finite-dimensional state spaces (FD-oscillators) and which become the usual boson oscillator when the dimension N of this space goes to infinity. In the present work we shall consider FD-oscillators which in the limit $N \to \infty$ become the generalized oscillator $\mathcal{A}_\mu$ described above. In consideration of FD-oscillators we shall use the following notations. We denote by $\mathcal{H}_\mu^{(N)}$ the $(N+1)$-dimensional subspace of the Hilbert space $\mathcal{F}_\mu (= \mathcal{H}_\mu)$ spanned by the first $N+1$ states $|0\rangle, |1\rangle, \ldots, |N\rangle$. In the space $\mathcal{H}_\mu^{(N)}$ the following orthogonality and completeness relations are fulfilled

$$\langle n|m \rangle = \delta_{n,m}, \qquad \sum_{n=1}^{N} |n\rangle\langle n| = \mathbb{1}_{N+1}. \tag{24}$$

The creation and annihilation operators in the space $\mathcal{H}_\mu^{(N)}$ are defined by relations

$$a_N^- := \sum_{k=1}^{N} \sqrt{2} b_{k-1} |k-1\rangle\langle k|, \tag{25}$$

$$a_N^+ := \sum_{k=1}^{N} \sqrt{2} b_{k-1} |k\rangle\langle k-1|. \tag{26}$$

They act on elements of basis in the natural way. Operators $\mathcal{N}_N$ and $B(\mathcal{N}_N)$ are defined in the same way as above. The commutation relations of operators $a_N^\pm, \mathcal{N}_N$ in space $\mathcal{H}_\mu^{(N)}$ look like

$$\begin{aligned} \left[a_N^-, a_N^+ \right] &= 2 \left(B(\mathcal{N}_N + \mathbb{1}_{N+1}) - B(\mathcal{N}_N) \right) - 2b_N^2 |N\rangle\langle N|, \\ \left[\mathcal{N}_N, a_N^\pm \right] &= \pm a_N^\pm. \end{aligned} \tag{27}$$

In the following section we shall describe the Krawtchouk oscillator as an important example of FD-oscillators.

2 Krawtchouk oscillator

1. Krawtchouk polynomials The Krawtchouk polynomials can be defined with help of the hypergeometric function (see [14])

$$K_n(x; p, N) := {}_2\widetilde{F}_1\left(\left.\begin{matrix} -n, -x \\ -N \end{matrix}\right| p^{-1}\right) = \sum_{k=0}^{N} \frac{(-n)_k(-x)_k}{k!(-N)_k p^k} \tag{28}$$

where $0 < p < 1$, $n = 0, 1, \ldots, N$, and where the Pochhammer symbol $(a)_k$ is defined by the relation

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdot\ldots\cdot(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

For definition of generalized oscillator it is convenient (following [4]) to redefine polynomials $K_n(x; p, N)$ as follows

$$\widetilde{K}_n(x; p, N) = \sqrt{\rho(n; p, N)}K_n(x; p, N), \quad n = 0, 1, \ldots, N, \tag{29}$$

where

$$\rho(n; p, N) = C_N^\xi p^\xi (1-p)^{N-\xi}, \qquad C_N^\xi := \frac{N!}{\Gamma(\xi+1)\Gamma(N-\xi+1)}. \tag{30}$$

The redefined polynomials satisfy to recurrent relations with symmetric Jacobi matrix ($n = 0, 1, \ldots, N$,)

$$x\widetilde{K}_n(x; p, N) = b_n\widetilde{K}_{n+1}(x; p, N) + a_n\widetilde{K}_n(x; p, N) + b_n\widetilde{K}_{n-1}(x; p, N), \tag{31}$$

$$\widetilde{K}_0(x; p, N) = 1, \tag{32}$$

where

$$a_n = p(N-n) + n(1-p), \qquad b_n = -\sqrt{p(1-p)(n+1)(N-n)}, \tag{33}$$

Together with polynomials $\widetilde{K}_n(x; p, N)$ we shall use their modified variant

$$\widehat{K}_n(x; p, N) = (-1)^n\widetilde{K}_n(x; p, N), , \quad n = 0, 1, \ldots, N. \tag{34}$$

The Krawtchouk polynomials $y(x) = K_n(x; p, N)$ satisfy the difference equation

$$-ny(x) = p(N-x)y(x+1) - [p(N-x) + x(1-p)]y(x) + x(1-p)y(x-1), \tag{35}$$

which remains valid also for polynomials $\widetilde{K}_n(x; p, N)$ and $\widehat{K}_n(x; p, N)$. The Krawtchouk polynomials fulfill the following orthogonality relations

$$\sum_{x=0}^{N} \rho(x; p, N)\widetilde{K}_m(x; p, N)\widetilde{K}_n(x; p, N) = \delta_{m,n}, \tag{36a}$$

$$\sum_{n=0}^{N} \rho(n; p, N)\widetilde{K}_n(x; p, N)\widetilde{K}_n(y; p, N) = \delta_{x,y}. \tag{36b}$$

2. The Krawtchouk oscillator. Now we define the Krawtchouk oscillator, according to general construction of oscillator-like system discussed above.

Let us denote by $\widetilde{\mathcal{H}}_{p,N} = \ell^2_{N+1}(\rho(x;p,N))$ the $N+1$-dimensional Hilbert space spanned by orthonormalized (with respect to weight function $\rho(x;p,N)$) basis $\left\{ \widetilde{K}_n(x;p,N) \right\}_{n=0}^{N}$ (by the symbol $\widehat{\mathcal{H}}_{p,N}$ we shall denote the same space in a case when polynomials $\left\{ \widehat{K}_n(x;p,N) \right\}_{n=0}^{N}$ are used as orthonormalized basis in this space).

Let's call the Krawtchouk oscillator the oscillator-like system determined by coordinate $\widetilde{X}$ and momentum $\widetilde{P}$ operators and quadratic Hamiltonian $\widetilde{H}$

$$\widetilde{X} := \mathrm{Re}(X - P), \quad \widetilde{P} := -i\mathrm{Im}(X - P), \quad \widetilde{H} := \frac{1}{4p(1-p)}\left(\widetilde{X}^2 + \widetilde{P}^2\right) \tag{37}$$

where operators X and P act on elements of basis $\left\{ \widetilde{K}_n(x;p,N) \right\}_{n=0}^{N}$ in the space $\widetilde{\mathcal{H}}_{p,N}$ according to formulas $(n \geq 1)$

$$X\widetilde{K}_0(x;p,N) = b_0\widetilde{K}_1(x;p,N) + a_0\widetilde{K}_0(x;p,N), \tag{38}$$

$$P\widetilde{K}_0(x;p,N) = -ib_0\widetilde{K}_1(x;p,N) + a_0\widetilde{K}_0(x;p,N), \tag{39}$$

$$X\widetilde{K}_n(x;p,N) = b_n\widetilde{K}_{n+1}(x;p,N) + a_n\widetilde{K}_n(x;p,N) + b_{n-1}\widetilde{K}_{n-1}(x;p,N), \tag{40}$$

$$P\widetilde{K}_n(x;p,N) = -ib_n\widetilde{K}_{n+1}(x;p,N) + a_n\widetilde{K}_n(x;p,N) + ib_{n-1}\widetilde{K}_{n-1}(x;p,N), \tag{41}$$

and coefficients a_n and b_n are determined by formulas (33). The creation and annihilation operators

$$\widetilde{a}^+ := \frac{1}{2\sqrt{p(1-p)}}\left(\widetilde{X} + i\widetilde{P}\right), \qquad \widetilde{a}^- := \frac{1}{2\sqrt{p(1-p)}}\left(\widetilde{X} - i\widetilde{P}\right) \tag{42}$$

act on basic elements of space $\widetilde{\mathcal{H}}_{p,N}$ according to formulas

$$\widetilde{a}^-\widetilde{K}_n(x;p,N) = -\sqrt{n(N-n+1)}\widetilde{K}_{n-1}(x;p,N), \tag{43}$$

$$\widetilde{a}^+\widetilde{K}_n(x;p,N) = -\sqrt{(n+1)(N-n)}\widetilde{K}_{n+1}(x;p,N). \tag{44}$$

These operators satisfy the commutation relations

$$\left[\widetilde{a}_K^-, \widetilde{a}_K^+\right] = (N - \mathbb{I} - 2\mathcal{N}) \tag{45}$$

where $\mathcal{N}$ - the operator numbering basis elements

$$\mathcal{N}\widetilde{K}_n(x;p,N) = n\widetilde{K}_n(x;p,N). \tag{46}$$

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Hamiltonian $\widetilde{H}$ can be written in the form

$$\widetilde{H} = \frac{1}{2}\left(\widetilde{a}^+\widetilde{a}^- + \widetilde{a}^-\widetilde{a}^+\right). \tag{47}$$

From the results of work [4] it follows, that this Hamiltonian has the spectrum

$$\lambda_n = N(n + \frac{1}{2}) - n^2, \quad (0 \le n \le N), \tag{48}$$

so that

$$\lambda_0 = \frac{1}{2}N = \lambda_N.$$

Analogously it is possible to define modified oscillator in the space $\widehat{\mathcal{H}}_{p,N}$ and check that these two oscillators are unitary equivalent (the creation and annihilation operators and Hamiltonian act identically on the appropriate bases).

Using results of work [5] it is possible to show that the eigenvalue equation $\widetilde{H}y = \lambda_n y$ in the space $\widetilde{\mathcal{H}}_{p,N}$ is equivalent to the difference equation (35). To receive an exact form of the Hamiltonian $\widetilde{H}$ in the space $\widetilde{\mathcal{H}}_{p,N}$ it is helpful to compare our definition of the Krawtchouk oscillator with the variant considered in [15] and [16].

3. Krawtchouk oscillator defined in [15]. In works [15] and [16] was considered another variant of the Krawtchouk oscillator with Hamiltonian

$$H_{AS}^K(\xi) = 2p(1-p)N + \frac{1}{2} + (1-2p)\frac{\xi}{h} - \sqrt{p(1-p)}\left[\alpha(\xi)e^{h\partial_\xi} + \alpha(\xi - h)e^{-h\partial_\xi}\right], \tag{49}$$

where

$$h = \sqrt{2Np(1-p)}, \qquad \alpha(\xi) = \sqrt{\left((1-p)N - \frac{\xi}{h}\right)\left(pN + 1 + \frac{\xi}{h}\right)}. \tag{50}$$

The operator H_{AS} is defined in the Hilbert space $\mathcal{H}_{AS} = \ell^2(\xi)$ with basis formed by Krawtchouk functions

$$\Psi_n(\xi) = (-1)^n\sqrt{C_N^n\left(\frac{p}{1-p}\right)^n \rho(pN + \frac{\xi}{h}; p, N)}\, K_n(pN + +1 + \frac{\xi}{h}); p, N), \tag{51}$$

which satisfy two (dual) orthogonality relations

$$\sum_{j=0}^{N}\Psi_n(\xi_j)\Psi_m(\xi_j) = \delta_{nm}, \qquad \sum_{j=0}^{N}\Psi_j(\xi_m)\Psi_j(\xi_n) = \delta_{nm}, \tag{52}$$

$\xi_j = h(j - pN), \quad j = 0, 1, \ldots, N$. These Krawtchouk functions are eigenfunctions for Hamiltonian H_{AS}

$$H_{AS}(\xi)\Psi_n(\xi) = \lambda_n\Psi_n(\xi); \qquad \lambda_n = n + \frac{1}{2}, \quad n = 0, 1, \ldots, N, \tag{53}$$

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which can be factorised

$$H_{AS}^K(\xi) = \frac{1}{2}\left[A^+, A^-\right] + \frac{1}{2}(N+1),$$

(54)

with the help of operators

$$A^+(\xi) = (1-p)e^{-h\partial_\xi}\alpha(\xi) - p\alpha(\xi)e^{h\partial_\xi} +$$
$$\sqrt{p(1-p)}\left((2p-1)N + \frac{2\xi}{h}\right),$$

(55)

$$A^-(\xi) = (1-p)\alpha(\xi)e^{h\partial_\xi} - pe^{-h\partial_\xi}\alpha(\xi) +$$
$$\sqrt{p(1-p)}\left((2p-1)N + \frac{2\xi}{h}\right).$$

(56)

These operators act on the elements of basis $\{\Psi_n(\xi)\}_{n=0}^N$ according to

$$A^+(\xi)\Psi_n^K(\xi) = \sqrt{(n+1)(N-n)}\,\Psi_{n+1}^K(\xi),$$
$$A^-(\xi)\Psi_n^K(\xi) = \sqrt{n(N-n+1)}\,\Psi_{n-1}^K(\xi).$$

(57)

Operators $A^+(\xi)$ and $A^-(\xi)$ together with the commutator

$$A_0(\xi) := \frac{1}{2}\left[A^+(\xi),\, A^-(\xi)\right]$$

(58)

satisfy the commutation relations of the $so(3)$ algebra

$$\left[A_0(\xi),\, A^\pm(\xi)\right] = \pm A^\pm(\xi), \quad \left[A^+(\xi),\, A^-(\xi)\right] = 2A_0(\xi).$$

(59)

4. Connection of two variants of Krawtchouk oscillator. In addition to considered above Hilbert spaces $\widetilde{\mathcal{H}}_{p,N} = \ell_{N+1}^2(\rho(x; p, N))$, $\widehat{\mathcal{H}}_{p,N}$ and $\mathcal{H}_{AS} = \ell^2(\xi)$ with bases $\left\{\widetilde{K}_n(x; p, N)\right\}_{n=0}^N$, $\left\{\widehat{K}_n(x; p, N)\right\}_{n=0}^N$ and $\{\Psi_n(\xi)\}_{n=0}^N$, respectively, we shall use auxiliary Hilbert space $\widehat{\mathcal{H}}_{AS} = \ell_{N+1}^2(\widetilde{\rho}(\xi))$ with basis $\left\{\widehat{K}_n(pN + \frac{\xi}{h}; p, N)\right\}_{n=0}^N$ orthogonal with respect to weight $\widetilde{\rho}(\xi) = \rho(pN + \frac{\xi}{h}; p, N)$.

Further we define unitary operators U, V and W, by the relations

$$U\widetilde{K}_n(x; p, N) = \widehat{K}_n(x; p, N), \quad U : \widetilde{\mathcal{H}}_{p,N} \to \widehat{\mathcal{H}}_{p,N}^K;$$

(60)

$$V\widehat{K}_n(pN + \frac{\xi}{h}; p, N) = \widetilde{\rho}(\xi)\widehat{K}_n(pN + \frac{\xi}{h}; p, N) = \Psi_n(\xi), \quad V : \widehat{\mathcal{H}}_{AS} \to \mathcal{H}_{AS};$$

(61)

$$W\widehat{K}_n(x; p, N) = \widehat{K}_n(pN + \frac{\xi}{h}; p, N), \quad W : \widehat{\mathcal{H}}_{p,N} \to \widehat{\mathcal{H}}_{AS}.$$

(62)

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Then the similarity transformation with the unitary operator

$$T := VWU, \qquad T : \widetilde{\mathcal{H}}_{p,N} \to \widehat{\mathcal{H}}_{AS} \tag{63}$$

realize unitary equivalence of the operator B_{AS} in $\mathcal{H}_{AS}$ with the operator $\widetilde{B}$ in $\widetilde{\mathcal{H}}_{p,N}$

$$\widetilde{B} = T^{-1} B_{AS} T. \tag{64}$$

Using the operators $\widetilde{a}^{\pm}$ (42) in space $\widetilde{\mathcal{H}}_{p,N}$, we define operators

$$\widetilde{K}_{\pm} = \frac{1}{\sqrt{2p(p-1)}} \widetilde{a}^{\pm}, \qquad \widetilde{K}_0 = \frac{1}{2} \left[\widetilde{K}_- \widetilde{K}_+ \right]. \tag{65}$$

These operators satisfy the commutation relations of the $so(3)$ Lie algebra

$$\left[\widetilde{K}_0, \, \widetilde{K}_{\pm} \right] = \pm \widetilde{K}_{\pm}, \qquad \left[\widetilde{K}_+, \, \widetilde{K}_- \right] = 2 \widetilde{K}_0 \tag{66}$$

and are unitary equivalent to the operators $A^{\pm}(\xi)$ and $A_0(\xi)$

$$A^{\pm}(\xi) = T \widetilde{K}_{\pm} T^{-1}, \qquad A_0(\xi) = T \widetilde{K}_0 T^{-1}. \tag{67}$$

Using the relations (67) and the explicit form (55)-(56) of operators $A^{\pm}(\xi)$, it is possible to find an explicit expressions for operators $\widetilde{K}_0$ and $\widetilde{K}_{\pm}$, and then for the basic operators $\widetilde{a}^{\pm}$ and $\widetilde{H}$ (and also operators $\widetilde{X}$ and $\widetilde{P}$) of our variant of Krawtchouk oscillator.

From (55), (56), and (61) we have

$$V^{-1} A^- V = \sqrt{p(1-p)} \left[\left((1-p)N + \frac{\xi}{h} \right) e^{h \partial_\xi} - \right.$$

$$\left. - \left(pN + \frac{\xi}{h} \right) e^{-h \partial_\xi} + \left((2p-1)N + \frac{2\xi}{h} \right) \right]; \tag{68}$$

$$V^{-1} A^+ V = (1-p) \sqrt{\frac{1-p}{p}} \left(pn + \frac{\xi}{h} \right) e^{-h \partial_\xi} -$$

$$- p \sqrt{\frac{p}{1-p}} \left((1-p)N - \frac{\xi}{h} \right) e^{h \partial_\xi} + \left((2p-1)N + \frac{2\xi}{h} \right); \tag{69}$$

$$V^{-1} H_{AS} V = 2p(1-p)N + \frac{1}{2} + (1-2p)\frac{\xi}{h} -$$

$$- \left[p \left((1-p)N - \frac{2\xi}{h} \right) e^{h \partial_\xi} + (1-p) \left(pN + \frac{2\xi}{h} \right) e^{-h \partial_\xi} \right]; \tag{70}$$

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$$V^{-1}A_0 V = V^{-1}H_{AS}V - \frac{1}{2}(N+1). \tag{71}$$

Then

$$W^{-1}\left(V^{-1}A^-V\right)W = \sqrt{p(1-p)}\left[(1-x)e^{\partial_x} - xe^{-\partial_x} + 2(x-N)\right]; \tag{72}$$

$$W^{-1}\left(V^{-1}A^-V\right)W = \sqrt{p(1-p)}\left[\frac{1-p}{p}xe^{-\partial_x} - \frac{p}{1-p}(1-x)e^{\partial_x} + (x-N)\right]; \tag{73}$$

$$W^{-1}\left(V^{-1}H_{AS}^K V\right)W = pN + \frac{x}{2} - 2px - \left(p(1-x)e^{\partial_x} + (1-p)xe^{-\partial_x}\right); \tag{74}$$

$$W^{-1}\left(V^{-1}A^0 V\right)W = W^{-1}\left(V^{-1}H_{AS}^K V\right)W - \frac{1}{2}(N+1), \tag{75}$$

and

$$T^{-1}A^- T = \sqrt{p(1-p)}\left((1-x)e^{\partial_x} - xe^{-\partial_x} + 2(x-N)\right); \tag{76}$$

$$T^{-1}A^+ T = \sqrt{p(1-p)}\left(\frac{1-p}{p}xe^{-\partial_x} - \frac{p}{1-p}(1-x)e^{\partial_x} + 2(x-N)\right); \tag{77}$$

$$T^{-1}H_{AS}^K T = pN + \frac{1}{2} + x - 2px - \left(p(1-x)e^{\partial_x} + (1-p)xe^{-\partial_x}\right); \tag{78}$$

$$T^{-1}A^0 T = T^{-1}H_{AS}^K T - \frac{1}{2}(N+1). \tag{79}$$

Because the selfadjoint operators $\widetilde{H}$ and $\widetilde{H}_{AS} = T^{-1}H_{AS}T$, acting in Hilbert space $\widetilde{\mathcal{H}}_{p,N}$, have the same set of eigenfunctions $\left\{\widetilde{K}_n(x;p,N)\right\}_{n=0}^{N}$ with corresponding eigenvalues

$$\lambda_n^{(1)} = \lambda_n(\widetilde{H}_K) = n + \frac{1}{2}, \quad \lambda_n^{(2)} = \lambda_n(\widetilde{H}_{AS}) = N(n+\frac{1}{2}) - n^2, \quad n = 0, 1, \ldots, N, \tag{80}$$

it follows that these Hamiltonians are connected by an interesting relation

$$\widetilde{H} = -(\widetilde{H}_{AS} - \frac{1}{2}\mathbb{1})^2 + N\widetilde{H}_{AS} = -\left(\widetilde{H}_{AS} - \frac{N+1}{2}\mathbb{1}\right)^2 + \frac{N(N+2)}{4}\mathbb{1} \tag{81}$$

or

$$\left(\widetilde{H} - \frac{N+1}{2}\mathbb{1}\right) + \left(\widetilde{H}_{AS} - \frac{N+1}{2}\mathbb{1}\right)^2 = \frac{N^2-2}{4}\mathbb{1}.$$

3 Coherent states for generalized oscillator in finite - dimensional Hilbert space

1. In the present section we define coherent states for generalized oscillator in finite-dimensional space $\mathcal{H}_\mu^{(N)}$ which can be obtained by the truncating procedure described above from appropriate

generalized oscillator in infinite dimensional space $\mathcal{H}_\mu$. In a limiting case when dimension N of the space $\mathcal{H}_\mu^{(N)}$ goes to infinity these coherent states become the coherent states for the generalized oscillator in $\mathcal{H}_\mu$.

We define coherent states in $\mathcal{H}_\mu^{(N)}$ by the relation

$$|z\rangle = \sum_{n=0}^{N} \frac{\left(z\widetilde{a}_N^+ - \bar{z}\widetilde{a}_N^-\right)^n}{n!}|0\rangle = \sum_{n=0}^{N}\left(\sqrt{2}b_{n-1}\right)!C_n(|z|)z^N|n\rangle, \tag{82}$$

where

$$C_n(|z|) = \sum_{m=0}^{\infty} \gamma_{n,m}\frac{(-|z|^2)^m}{(n+2m)!} \tag{83}$$

Multipliers $\gamma_{n,m}$ are defined from equality

$$\gamma_{n,m} = \sum_{k_1=0}^{n} 2b_{k_1}^2 \sum_{k_2=0}^{k_1+1} 2b_{k_2}^2 \cdots \sum_{k_m=0}^{k_{m-1}+1}, \tag{84}$$

where b_k - coefficients from recurrent relations (18) under an additional condition, that $b_m = 0$ at $m \geq N$.

Multipliers $\gamma_{n,m}$ satisfy to recurrent relations

$$\gamma_{n+1,m} - \vartheta_{n+1}\gamma_{n,m} = 2b_{n+1}^2\vartheta_{n+2}\gamma_{n+2,m-1};$$
$$\gamma_{n,0} = 1, \quad \gamma_{0,m} = 2b_0^2\gamma_{1,m-1}, \tag{85}$$

where

$$\vartheta_n = \begin{cases} 1 & \text{if} \quad n \leq N \\ 0 & \text{if} \quad n > N \end{cases}. \tag{86}$$

Coherent states (82) can be written down in the form

$$|z\rangle = \sum_{l=0}^{N}\sum_{n=l}^{\infty} d_{n,l}^N\frac{\left(\sqrt{2}b_{l-1}\right)!}{n!}(-\bar{z})^{\frac{1}{2}(n-l)}z^{\frac{1}{2}(n+l)}|l\rangle, \tag{87}$$

which close to the expression for coherent states of standard boson FD-oscillator, considered in [18]-[19]. Coefficients $d_{n,l}^N$ satisfy the recurrent relations

$$d_{n,l}^N = \vartheta_l d_{n-1,l-1}^N + 2b_l^2\vartheta_{l+1}d_{n-1,l+1}^N, \tag{88}$$

with boundary conditions

$$d_{n-1,-1}^N = 0, \qquad d_{0,0}^N = 1, \qquad d_{n,n+k}^N = 0, \qquad k > 0. \tag{89}$$

It is easy to prove (compare [18]-[19]), that the solution of relations (88), (89) looks like

$$d_{n,l}^N = \frac{C_N}{(2b_l^2)!} \sum_{l=0}^{N} \frac{\widetilde{\psi}_l(x_k)}{\left(\widetilde{\psi}_N(x_k)\right)^2} x_k^n, \tag{90}$$

where

$$\widetilde{\psi}_l(\sqrt{2}x) = (\sqrt{2}b_{l-1})! \psi_l^{(0)}(x) \tag{91}$$

and x_k are roots of the equation

$$\widetilde{\psi}_{N+1}(x_k) = 0, \quad k = 0, 1, \ldots, N. \tag{92}$$

Let us remind that recurrent relations (20) for polynomials $\psi_n^{(0)}(x)$ differ from relations (18) for polinomials $\psi_n(x)$ by absence of diagonal members, i.e. $a_n = 0$. The constant C_N in equality (90) can be determined from a normalization condition. Recurrent relations for polynomials $\widetilde{\psi}_n(x)$ look like

$$x\widetilde{\psi}_n(x) = \widetilde{\psi}_{n+1}(x) + 2b_{n-1}^2 \widetilde{\psi}_{n-1}(x). \tag{93}$$

2. Now we calculate a value of the constant C_N in the case of generalized oscillator. (This expression reproduces the value of C_N for standard boson FD-oscillator, given in [18]-[19] without the proof). To this end we rewrite the formula (87) as

$$|z\rangle = \sum_{l=0}^{N} C_l^{(N)} |l\rangle, \tag{94}$$

where

$$C_l^{(N)} = \sum_{n=l}^{\infty} \frac{(\sqrt{2}b_{l-1})!}{n!} d_{n,l}^N (-\bar{z})^{\frac{1}{2}(n-l)} z^{\frac{1}{2}(n+l)}, \tag{95}$$

and coefficients $d_{n,l}^N$ are determined by the relation (90). Then the normalization condition from which we determine C_N takes the form

$$\sum_{l=0}^{N} \left| C_l^{(N)} \right|^2 = 1. \tag{96}$$

In view of (90) and the last condition from (89) the relation (95) can be rewritten as

$$C_l^{(N)} = \frac{C_N}{(\sqrt{2}b_{l-1})!} \left(-\frac{iz}{|z|}\right)^l \sum_{k=0}^{N} \frac{\widetilde{\psi}_l(x_k)}{\left(\widetilde{\psi}_N(x_k)\right)^2} e^{i|z|x_k}. \tag{97}$$

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Then

$$\left| C_l^{(N)} \right|^2 = \frac{C_N^2}{(2b_{l-1}^2)!} \left| A_{l,N}(|z|) \right|^2, \tag{98}$$

where

$$A_{l,N}(|z|) = \sum_{k=0}^{N} \frac{\widetilde{\psi}_l(x_k)}{\left(\widetilde{\psi}_N(x_k) \right)^2} e^{i|z|x_k}. \tag{99}$$

Let's check the validity of the relation

$$\frac{\mathrm{d}}{\mathrm{d}|z|} \left| A_{l,N}(|z|) \right|^2 = 0. \tag{100}$$

From (99) we have

$$\frac{\mathrm{d}}{\mathrm{d}|z|} \left| A_{l,N}(|z|) \right|^2 = B_{l,N}(|z|)\overline{A_{l,N}(|z|)} + \overline{B_{l,N}(|z|)} A_{l,N}(|z|), \tag{101}$$

$$B_{l,N}(|z|) = \left(A_{l,N}(|z|) \right)'_{|z|} = \sum_{k=0}^{N} \frac{\widetilde{\psi}_l(x_k)}{\left(\widetilde{\psi}_N(x_k) \right)^2} i x_k \, e^{i|z|x_k}. \tag{102}$$

Because the summands in the RHS of (101) are complex conjugate to each other, we have

$$\frac{\mathrm{d}}{\mathrm{d}|z|} \left(\sum_{l=0}^{N} \frac{|A_{l,N}(|z|)|^2}{(2b_l^2)!} \right) = \left(\sum_{l=0}^{N} \frac{2\mathrm{Re}\left[A_{l,N}(|z|)\overline{B_{l,N}(|z|)} \right]}{(2b_l^2)!} \right). \tag{103}$$

From (91) and recurrent relations (20) it follows that zeros $x_0, x_1, \ldots, x_N$ of the polynomial (92) are located symmetrically concerning the beginning of coordinates. Then for $l = 2p$ $\widetilde{\psi}_l(x_k) = \widetilde{\psi}_l(x_{N-k})$ and coefficients at $e^{i|\alpha|x_k}$ and $e^{-i|\alpha|x_k}$ in $A_{l,N}(|z|)$ are equal. For the case $l = 2p + 1$ we have $\widetilde{\psi}_l(x_k) = -\widetilde{\psi}_l(x_{N-k})$ and coefficients at $e^{i|\alpha|x_k}$ and at $e^{-i|\alpha|x_k}$ in $A_{l,N}(|z|)$ have equals module and are opposite on the sign. Using notation

$$a_{l,k}^{(N)} = \frac{\widetilde{\psi}_l(x_k)}{\widetilde{\psi}_N^2(x_k)}, \qquad\qquad b_{l,k}^{(N)} = a_{l,k}^{(N)} x_k; \tag{104}$$

$$A_{l,N} = \sum_{k=0}^{N} a_{l,k}^{(N)} e^{i|z|x_k}, \qquad\qquad B_{l,N} = i \sum_{k=0}^{N} b_{l,k}^{(N)} e^{i|z|x_k}, \tag{105}$$

and taking into account above mentioned reasoning we obtain

$$
A_{l,N} =
\begin{cases}
2\sum_{k=0}^{m-1} a_{2p,k}^{(N)} \cos(|z|x_k) + a_{2p,m}^{(N)}, & N = 2m,\ l = 2p; \\[2ex]
2\sum_{k=0}^{m} a_{2p,k}^{(N)} \cos(|z|x_k), & N = 2m+1,\ l = 2p; \\[2ex]
2i\sum_{k=0}^{m-1} a_{2p+1,k}^{(N)} \sin(|z|x_k) + a_{2p+1,m}^{(N)}, & N = 2m,\ l = 2p+1; \\[2ex]
2i\sum_{k=0}^{m} a_{2p+1,k}^{(N)} \sin(|z|x_k), & N = 2m+1,\ l = 2p+1;
\end{cases}
\tag{106}
$$

$$
\overline{B_{l,N}} =
\begin{cases}
-2\sum_{k=0}^{m-1} b_{2p,k}^{(N)} \sin(|z|x_k) + b_{2p,m}^{(N)}, & N = 2m,\ l = 2p; \\[2ex]
-2\sum_{k=0}^{m} b_{2p,k}^{(N)} \sin(|z|x_k), & N = 2m+1,\ l = 2p; \\[2ex]
-2i\sum_{k=0}^{m-1} b_{2p+1,k}^{(N)} \cos(|z|x_k) + b_{2p+1,m}^{(N)}, & N = 2m,\ l = 2p+1; \\[2ex]
-2i\sum_{k=0}^{m} b_{2p+1,k}^{(N)} \cos(|z|x_k), & N = 2m+1,\ l = 2p+1;
\end{cases}
\tag{107}
$$

Note that for obtaining (106) and (107), we used relations

$$
\begin{gathered}
a_{2p,k}^{(N)} = a_{2p,N-k}^{(N)}, \qquad a_{2p+1,k}^{(N)} = -a_{2p+1,N-k}^{(N)}, \\
b_{2p,k}^{(N)} = -b_{2p,N-k}^{(N)}, \qquad b_{2p+1,k}^{(N)} = b_{2p+1,N-k}^{(N)}, \\
b_{l,m}^{2m)} = 0,
\end{gathered}
\tag{108}
$$

following from definitions (104) and (105). From (106), (107) and (3) we receive

$$
\mathrm{Re}\left(A_{l,N}(|z|)\overline{B_{l,N}(|z|)}\right) =
$$

$$= \begin{cases} -4\sum_{s,k=0}^{m-1} a_{2p,s}^{(N)}b_{2p,k}^{(N)}\cos(|z|x_s)\sin(|z|x_k)- \\ \qquad\qquad -2\sum_{k=0}^{m-1} a_{2p,m}^{(N)}b_{2p,k}^{(N)}\sin(|z|x_k), \quad N=2m,\ l=2p; \\ -4\sum_{s,k=0}^{m} a_{2p,s}^{(N)}b_{2p,k}^{(N)}\cos(|z|x_s)\sin(|z|x_k), \quad N=2m+1,\ l=2p; \\ 4\sum_{s,k=0}^{m-1} a_{2p+1,s}^{(N)}b_{2p+1,k}^{(N)}\cos(|z|x_k)\sin(|z|x_s) \quad N=2m,\ l=2p+1; \\ 4\sum_{s,k=0}^{m} a_{2p+1,s}^{(N)}b_{2p+1,k}^{(N)}\cos(|z|x_k)\sin(|z|x_s), \quad N=2m+1,\ l=2p; \end{cases} \quad (109)$$

Substituting (109) in the right-hand side of (103) we see, that it is sufficient for validity of equality (100) that the coefficients at functions $\cos(|z|x_s)\sin(|z|x_k)$ $(k,s=0,1,\ldots,(m-1)$, if $N=2m$ and $k,s=0,1,\ldots,m$, if $n=2m+1$, $x_s\neq 0$, $x_k\neq 0$,) vanish, i.e.

$$\sum_{l=0}^{N}(1-2\delta_{s,k})^l\frac{\widetilde{\psi}_l(x_s)\widetilde{\psi}_l(x_k)}{(2b_l{}^2)!}x_k=0, \quad (110)$$

and also the coefficients at functions $\sin(|z|x_k)$ $(k=0,1,\ldots,(m-1)$, if $N=2m$, $l=2p$, $x_k\neq 0$,) vanish, i.e.

$$\sum_{p=0}^{m}\frac{\widetilde{\psi}_{2p}(0)\widetilde{\psi}_{2p}(x_k)}{(2b_l{}^2)!}x_k=0, \quad (111)$$

Now we prove the relation (110). For $0\neq x_k\neq x_s\neq 0$ the relation (110) is equivalent to equality

$$L=x_s x_k\sum_{l=0}^{N}\frac{\widetilde{\psi}_l(x_s)\widetilde{\psi}_l(x_k)}{(2b_{l-1}^2)!}=0, \quad (112)$$

For the proof of this equality we apply to $x_s\widetilde{\psi}_l(x_s)$ the recurrent relation (93). The result is

$$L=x_k\left(\sum_{l=0}^{N-1}\frac{\widetilde{\psi}_{l+1}(x_s)\widetilde{\psi}_l(x_k)}{(2b_{l-1}^2)!}+\sum_{l=1}^{N}\frac{\widetilde{\psi}_{l-1}(x_s)\widetilde{\psi}_l(x_k)}{(2b_{l-2}^2)!}\right)$$

If we replace l by $(l+1)$ in the second sum we get

$$L=x_k\sum_{l=0}^{N-1}\frac{\widetilde{\psi}_{l+1}(x_s)\widetilde{\psi}_l(x_k)+\widetilde{\psi}_l(x_s)\widetilde{\psi}_{l+1}(x_k)}{(2b_{l-1}^2)!}=x_k\Pi_N.$$

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Similarly, applying the recurrent relation (93) to $x_k \widetilde{\psi}_l(x_k)$, we obtain the equality

$$L = x_k \Pi_N = x_s \Pi_N.$$

Because $0 \neq x_k \neq x_s \neq 0$, this relation means $\Pi_N = 0$ and therefore equality (112) is valid.

Now we consider the case $x_k = x_s \neq 0$. We rewrite (110) as

$$\sum_{l=0}^{N} (-1)^l \frac{\widetilde{\psi}_l^2(x_k)}{\left(2b_{l-1}^2\right)!} x_k = 0. \tag{113}$$

Using the recurrent relation (93), by induction we obtain

$$x \sum_{l=0}^{N} (-1)^l \frac{\widetilde{\psi}_l(x)}{\left(2b_{l-1}^2\right)!} = (-1)^N \frac{\widetilde{\psi}_{N+1}(x)\widetilde{\psi}_N(x)}{\left(2b_{N-1}^2\right)!}, \tag{114}$$

from which it follows (taking into account, that x_k are roots of function $\widetilde{\psi}_{N+1}(x)$) the relation (113).

Now we prove (111). It is easily to check that

$$\widetilde{\psi}_{2p}(0) = (-1)^p \left(2b_{2p}^2\right)!!. \tag{115}$$

Substituting this expression in (111) we see that (111) is equivalent to relation

$$\sum_{p=0}^{m} (-1)^p \frac{\widetilde{\psi}_{2p}(x_k)}{\left(2b_{2p-1}^2\right)!!} = 0, \tag{116}$$

where $\widetilde{\psi}_{2m+1}(x_k) = 0$ and $x_k \neq 0$. Using (93) by induction it is possible to prove identity

$$x \sum_{p=0}^{m} (-1)^p \frac{\widetilde{\psi}_{2p}(x)}{\left(2b_{2p-1}^2\right)!!} = (-1)^m \frac{\widetilde{\psi}_{2m+1}(x)}{\left(2b_{2m-1}^2\right)!!}, \tag{117}$$

from which at $x = x_k$ the relation (116) follows.

From (100) it follows, that the normalization condition (96) can be considered for $|z| = 0$. Because from (89), (90) and (99) it follows that

$$|A_{l,N}(0)| = \left| \sum_{k=0}^{N} \frac{\widetilde{\psi}_l'(x_k)}{\widetilde{\psi}_N^2(x_k)} \right| = \frac{2b_{l-1}^2}{C_N} \left| d_{0,l}^{(N)} \right| = 0 \qquad l > 0, \tag{118}$$

we obtain

$$A_{0,N}(0) = C_N \sum_{k=0}^{N} \frac{1}{\widetilde{\psi}_N^2(x_k)}. \tag{119}$$

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Further, from (98), (118) and (119) we have

$$\left|C_l^{(N)}(0)\right| = 0, \qquad l > 0 : \left|C_0^{(N)}\right| = C_N \sum_{k=0}^{N} \frac{1}{\widetilde{\psi}_N^2(x_k)}. \tag{120}$$

Substituting the relation (120) in the normalization condition (106) we obtain

$$C_N = \left(\sum_{k=0}^{N} \left[\widetilde{\psi}_N^2(x_k) \right]^{-2} \right)^{-1}. \tag{121}$$

This relation is a generalization of the appropriate formula for a normalizing constant from [19] which is given there without the proof. Therefore we shall give bellow the proof of the expression for this constant also for the case of Hermite polynomials $\mathrm{He}_n(x)$ considered in [19].

<u>**3.**</u> Let us calculate expression in the right-hand side of (121) in a case of polynomials $\widetilde{\psi}_n(x) = \mathrm{He}_n(x)$, which fulfill recurrent relations

$$\mathrm{He}_{n+1}(x) = x\mathrm{He}_n(x) - n\mathrm{He}_{n-1}(x), \quad \mathrm{He}_0(x) = 1; \tag{122}$$
$$(\mathrm{He}_{n+1}(x))'_x = (n+1)\mathrm{He}_n(x) \tag{123}$$

We consider Lagrange interpolation polynomial $P_N(x)$ for a polynomial $\mathrm{He}_n(x)$ for interpolation points $x_0^{N+1}, x_1^{N+1}, \ldots, x_N^{N+1}$ which are roots of a polynomial $\mathrm{He}_{n+1}(x)$, i.e.

$$\mathrm{He}_{n+1}(x_k^{N+1}) = 0, \quad k = 0, 1, \ldots, N, \tag{124}$$

and the values of which in these points are equal to

$$P_N(x_k^{N+1}) = \left[\mathrm{He}_N(x_k^{N+1}) \right]^{-1}, \quad k = 0, 1, \ldots, N. \tag{125}$$

It is known [22] that

$$P_N(x) = \sum_{k=0}^{N} \frac{\mathrm{He}_{N+1}(x)}{(x - x_k^{N+1})\mathrm{He}_N(x_k^{N+1})\mathrm{He}'_{N+1}(x_k^{N+1})} =$$
$$= \frac{1}{N+1} \sum_{k=0}^{N} \frac{\mathrm{He}_{N+1}(x)}{(x - x_k^{N+1})\mathrm{He}_N^2(x_k^{N+1})}. \tag{126}$$

From (126) and (121) we see that C_N^{-1} coincides with the leading coefficient of the polynomial $(N+1)P_N(x)$. We introduce an auxiliary polynomial

$$\Phi_{2N}(x) = P_N(x)\mathrm{He}_N(x) - 1, \tag{127}$$

leading coefficient of which is equal to

$$K_N = \frac{1}{(N+1)C_N}. \tag{128}$$

From (125) and (127) it follows that $\Phi_{2N}(x_k^{N+1}) = 0, \quad k = 0, 1, \ldots, N$, so the polynomial $\Phi_{2N}(x)$ can be divided on the polynomial $\mathrm{He}_{N+1}(x)$

$$\Phi_{2N}(x) = \mathrm{He}_{N+1}(x)Q_{N-1}(x), \tag{129}$$

where $Q_{N-1}(x)$ is a polynomial of the order $N-1$. Because from (122) it follows that the leading coefficient of a polynomial $\mathrm{He}_N(x)$ is equal to 1, we see from (128) and (129) that the leading coefficients of a polynomial $Q_{N-1}(x)$ is equal to K_N. Further, from (127) and (129) we have

$$\mathrm{He}_{N+1}(x_k^N)Q_{N-1}(x_k^N) = -1, \quad k = 0, 1, \ldots, N-1, \tag{130}$$

where x_k^N are roots of the polynomial $\mathrm{He}_N(x)$. Taking into account (122), we have

$$\mathrm{He}_{N+1}(x_k^N) = -N\mathrm{He}_{N-1}(x_k^N), \quad k = 0, 1, \ldots, N-1, \tag{131}$$

Substituting (131) in (130) and taking into account (125), we obtain

$$Q_{N-1}(x_k^N) = \frac{1}{N\mathrm{He}_{N-1}(x_k^N)} = \frac{1}{P_{N-1}(x_k^N)}, \quad k = 0, 1, \ldots, N-1. \tag{132}$$

Because two polynomials $Q_{N-1}(x)$ and $P_{N-1}(x)$, coinciding in N points $x_0^N, x_1^N, \ldots, x_{N-1}^N$, are equal, the leading coefficients of a polynomial $P_{N-1}(x)$ is equal to

$$N\,K_N = \frac{N}{(N+1)C_N}. \tag{133}$$

Taking into account that $P_0(x) = 1$ and continuing by induction, we received

$$C_N = \frac{N!}{N+1}. \tag{134}$$

Returning to the general case, we receive, taking into account (94), (97) and (121), an analytical expression for coherent states for the generalized oscillator

$$|z\rangle = \left(\sum_{k=0}^N \left[\widetilde{\psi}_N(x_k) \right]^{-2} \right)^{-1} \sum_{l=0}^N \frac{\left(-i\frac{z}{|z|} \right)^l}{(\sqrt{2}b_{l-i})!} \left(\sum_{k=0}^N \frac{\widetilde{\psi}_l(x_k)}{\widetilde{\psi}_N^2(x_k)} e^{i|z|x_k} \right) |l\rangle, \tag{135}$$

where x_k are roots of the equation (92).

4. The completeness for coherent states (135) can be proved in the standard way. For To construct a measure

$$d\mu(|z|^2) = \widetilde{W}(|z|^2)d^2z \tag{136}$$

participating in the decomposition of unit

$$\iint_{\mathbb{C}} \widetilde{W}(|z|^2)|z\rangle\langle z|d^2z = \mathbb{1}, \tag{137}$$

where $d^2z = d\mathrm{Re}zd\mathrm{Im}z$, we designate $x = |z|^2$ and substitute (135) in (137). Using (99) and (121), we obtain

$$\sum_{l=0}^{N} \frac{\pi C_N^2}{(2b_{l-1}^2)!} \int_0^\infty \widetilde{W}(x) \left|A_{l,N}(\sqrt{x})\right|^2 dx|l\rangle\langle l| = 1. \tag{138}$$

Using the notation

$$W(x) = \pi\widetilde{W}(x) \tag{139}$$

we obtain the following condition

$$\int_0^\infty W(x)|A_{l,N}|^2dx = (2b_{l-1}^2)!, \quad l = 0, 1, \ldots, N \tag{140}$$

for definition of a function $W(x)$ We shall not discuss here the solution of this (trigonometrical) moments problem [23].

5. Coherent states for Krawtchouk oscillator can be obtained from the general formula (135) if we replace functions $\widetilde{\psi}_n(x)$ by the Krawtchouk polynomials with recurrent relations

$$x\widetilde{\psi}_n(x) = \widetilde{\psi}_{n+1}(x) + 2p(1-p)n(N-n+1)\widetilde{\psi}_{n-1}(x), \quad \widetilde{\psi}_0(x) = 1, \tag{141}$$

Note, that coherent states obtained in this way differ from "spin" coherent states for Krawtchouk oscillator given in [15] as well as from "phase" coherent states defined in the work [24].

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